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ADDENDUM TO "EXAMPLES OF SEMIPERFECT RINGS"*

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ABSTRACT

We fill a gap in [4], and provide a rigorous example of a local ring R whose Jacobson radical is locally nilpotent, but $M_2(R)$ is not strongly π-regular.

Recall that a ring R is called **left π-regular** if it satisfies the DCC on chains of the form

$$Rr \supset Rr^2 \supset Rr^3 \cdots .$$

Although the terminology is unfortunate, this condition arises naturally in the study of the Krull–Schmidt decompositions for finitely presented modules, cf. [3]; in fact, a necessary and sufficient condition for this theory to hold is for the matrix ring $M_n(R)$ to satisfy this property, for all n. Thus, one was led to the question of whether R left π-regular implies $M_n(R)$ is left π-regular. Dischinger observed that a positive answer would imply Koethe's conjecture [1, p. 572; 2, pp. 47, 48]. However, in [4] a theory was developed for determining whether or

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not an algebra is strongly π-regular, and counterexamples were given on the basis of these conditions.

Unfortunately the conditions on [4, example 2.3, p. 278] imply $A = B = 0$, so the example does not work as stated. This could be corrected by letting I be the ideal of T generated by:

$$A^2, \quad B^2, \quad AwA, \quad AwB, \quad BwB$$

for all w in $\mathcal{B} \setminus \{1\}$. However, it is now unclear whether or not J is nil. In fact, it now looks as though this example may be of difficulty comparable to Koethe's conjecture. On the other hand, [4, example 2.5] can be salvaged through a more careful analysis.

The point of this note is to study the precise set-up of [4, examples 2.3 and 2.5], in order to provide a rigorous counterexample, and also to obtain more precise information. The Jacobson radical of $M_n(R)$ remains nil, even though $M_n(R)$ is not left π-regular.

Example 1: Based on [4, Example 2.5]. A local ring R_0 whose Jacobson radical is locally nilpotent, but $M_2(R_0)$ is not strongly π-regular.

Let F be a field. Let $F(X)$ be the field of fractions of the polynomial ring $F[X]$, and extend $\{X^i : i \in \mathbb{Z}\}$ to a basis $\mathcal{B}$ of $F(X)$ over F. Let T be the free product of $F(X)$ with the free algebra $F\langle A, B\rangle$.

Let $V = \{BwA : w \in \mathcal{B} \setminus \{X^{-i} : i \text{ is a positive integer}\}\}$, and let P be the ideal of T generated by

$$A^2, B^2, AwA, AwB, BwB, \qquad \text{for all } w \in \mathcal{B} \setminus \{1\}, V, \qquad \text{and}$$
$$\bigcup_{k=1}^{\infty} \{BX^{-1}A, \ldots, BX^{-k}A\}^{n_k}, \qquad \text{where } n_k > 2^k + 1.$$

Let $R_0 = T/P$, and let a, b, x, z_i denote the respective images in R_0 of $A, B, X, BX^{-i}A$ under the natural projection. In particular $bx^i a = 0$ for all $i \geq 0$. It is clear that R_0 is graded by total degree in a and b. We denote by $\deg(s)$ the degree of $s \in R_0$. Define $J = R_0aR_0 + R_0bR_0$, a graded ideal. Clearly J is a maximal ideal of R_0 since $R_0/J \approx F(x)$.

Note that the $BX^{-i}A$ generate a free subalgebra of T. Since $\deg(z_i) = 2$, it is easy to see that if $z_{i_1} \cdots z_{i_m} = z_{j_1} \cdots z_{j_n} \neq 0$ then $n = m$ and $i_k = j_k$ for $1 \leq k \leq m$. Thus the nonzero elements of the form $z_{i_1} \cdots z_{i_m}$ form a basis $\mathcal{B}_1$ of $F\langle z_1, \ldots, z_n, \ldots \rangle$.

Note also that we may identify $F(X)$ with $F(x)$, so $\mathcal{B}$ is thereby viewed as a basis of $F(x)$.

Remark 2: $bR_0a = \sum_{i=1}^{\infty} z_iF\langle z_1, \ldots, z_n, \ldots\rangle$, and bR_0a is locally nilpotent since $\{z_1, \ldots, z_k\}^{n_k} = \{0\}$.

LEMMA 3: *For each integer $m \geq 3$ there exist $z_{j_1}, \ldots, z_{j_{2^{m-1}+1}}$ such that*

$$\begin{aligned}0 \neq & z_{j_1} \cdots z_{j_{2^{m-1}+1}} \\ =& bx^{-1}(x^{i_1}abx^{-i_1-2})(x^{i_2}abx^{-i_2-2}) \cdots (x^{i_{2^{m-1}}}abx^{-i_{2^{m-1}}-2})x^m a,\end{aligned}$$

for suitable $i_1 = 0,\ i_1, \ldots,\ i_{2^{m-1}-1},\ i_{2^{m-1}} = m-1$, where

$$1 \leq i_2, \ldots, i_{2^{m-1}-1} \leq m-2.$$

Proof: By induction on m. For $m = 3$ we have

$$bx^{-1}(abx^{-2})(xabx^{-3})(xabx^{-3})(x^2abx^{-4})x^3a = z_1z_1z_2z_1z_1 \neq 0$$

(taking $i_1 = 0,\ i_2 = i_3 = 1,\ i_4 = 2$), because $2 < n_1$ and $5 < n_2$. For $m \geq 3$ assume we have

$$0 \neq z_{j_1} \cdots z_{j_{2^{m-1}+1}} = bx^{-1}(x^{i_1}abx^{-i_1-2}) \cdots (x^{i_{2^{m-1}}}abx^{-i_{2^{m-1}}-2})x^m a,$$

where $i_1 = 0,\ i_{2^{m-1}} = m-1$ and $1 \leq i_2, \ldots, i_{2^{n-1}-1} \leq m-2$. One sees easily that $z_{j_1}, \ldots, z_{j_{2^{m-1}+1}} \in \{z_1, \ldots, z_{m-1}\}$ and

$$z_{j_1} \cdots z_{j_{2^{m-1}}} z_m z_{j_2} \cdots z_{j_{2^{m-1}+1}} \neq 0.$$

Also we have

$$\begin{aligned}&z_{j_1} \cdots z_{j_{2^{m-1}}} z_m z_{j_2} \cdots z_{j_{2^{m-1}+1}} \\ &= bx^{-1}(x^{i_1}abx^{-i_1-2}) \cdots (x^{i_{2^{m-1}}}abx^{-i_{2^{m-1}}-2})(x^{i_1+1}abx^{-i_1-3}) \cdots \\ &\quad \cdots (x^{i_{2^{m-1}}+1}abx^{-i_{2^{m-1}}-3})x^{m+1}a,\end{aligned}$$

so the result follows. ∎

LEMMA 4: *$M_2(R_0)$ is not strongly π-regular.*

Proof: Let $r = \begin{pmatrix} x & a \\ b & 0 \end{pmatrix}$. Then by induction on n it is easy to see that

$$r^n = \begin{pmatrix} x(n) & a(n) \\ b(n) & 0 \end{pmatrix},$$

where

(i) $x(n) = x^n + \sum_{i=0}^{n-2} x^i abx^{n-i-2}$,
(ii) $a(n) = x^{n-1}a$,
(iii) $b(n) = bx^{n-1}$.

Suppose that there exist $\alpha, \beta, \gamma, \delta \in R_0$ such that

$$r^n = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} r^{2n}$$

for some $n > 3$. Comparing coefficients of the bottom row yields

$$b(n) = \gamma(x(n)^2 + a(n)b(n)) + \delta b(n)x(n)$$

and

$$0 = \gamma x(n)a(n).$$

Thus

$$\begin{aligned} b(n)x(n)^{-1}a(n) &= (\gamma(x(n)^2 + a(n)b(n)) + \delta b(n)x(n))x(n)^{-1}a(n) \\ &= \gamma a(n)b(n)x(n)^{-1}a(n) + \delta b(n)a(n) \\ &= \gamma a(n)b(n)x(n)^{-1}a(n), \end{aligned}$$

since $b(n)a(n) = bx^{2(n-1)}a = 0$. Hence

$$(1 - \gamma a(n))b(n)x(n)^{-1}a(n) = 0.$$

Since R_0 is graded, $1 - \gamma a(n)$ cannot be a zero divisor. (Indeed, we shall see that it is invertible.) Hence $b(n)x(n)^{-1}a(n) = 0$.

By definition of P,

$$\left(\sum_{i=0}^{n-2} x^i abx^{-i-2}\right)^{n_n+1} = 0.$$

Hence

$$b(n)x(n)^{-1}a(n) =$$
$$bx^{-1}\left(1 - \sum_{i=0}^{n-2} x^i abx^{-i-2} + \cdots + (-1)^{n_n}(\sum_{i=0}^{n-2} x^i abx^{-i-2})^{n_n}\right) x^{n-1}a = 0.$$

Taking homogeneous parts in a and b, we see that

$$bx^{-1}\left(\sum_{i=0}^{n-2} x^i abx^{-i-2}\right)^m x^{n-1}a = 0,$$

for all $0 \leq m \leq n_n$. This contradicts Lemma 3, since $n > 3$. Thus $M_2(R_0)r^n \neq M_2(R_0)r^{n+1}$ for all positive integers n. Therefore $M_2(R_0)$ is not strongly π-regular. ∎

Lemma 4 shows the existence of nonzero good terms in [4, example 2.5]. To prove in Example 1 that R_0 is indeed local, with its maximal ideal J locally nilpotent, we use the following result.

MAIN THEOREM 5: *Let F be a field. Let R be an F-algebra and let $D \subset R$ be a division ring. Suppose that there exists an F-basis $\mathcal{B}$ of D with $1 \in \mathcal{B}$, and two elements $a, b \in R \setminus D$ such that D, a, b generate R as F-algebra, and furthermore $a^2, b^2, ba, awa, awb, bwb = 0$ for all $w \in \mathcal{B} \setminus \{1\}$.*

If bRa is locally nilpotent, then so is the ideal $RaR + RbR$.

Proof: Let $T = bRa$ and $I = RaR + RbR$. Clearly

$$I = DaD + DbD + DabD + DTD + DaTD + DTbD + DaTbD,$$

so $I^3 \subseteq DTD + DaTD + DTbD + DaTbD$. Let $p_1, \ldots, p_r \in I^3$. Then there exist a positive integer n, and elements $t_{ik} \in T$, $\delta_{ik}, \delta'_{ik} \in D$ and $\epsilon_{ik}, \epsilon'_{ik} \in \{0, 1\}$ such that

$$p_i = \sum_{k=1}^{n} \delta_{ik} a^{\epsilon_{ik}} t_{ik} b^{\epsilon'_{ik}} \delta'_{ik}.$$

Let us study the product $t_{ik} b^{\epsilon'_{ik}} \delta'_{ik} \delta_{jl} a^{\epsilon_{jl}} t_{jl}$.

(1) If $\epsilon'_{ik}, \epsilon_{jl} = 0$, then $t_{ik}\delta'_{ik}\delta_{jl}t_{jl} = \alpha_{ikjl} t_{ik} t_{jl}$ for some $\alpha_{ikjl} \in F$.

(2) If $\epsilon'_{ik} + \epsilon_{jl} = 1$, then $t_{ik} b^{\epsilon'_{ik}} \delta'_{ik} \delta_{jl} a^{\epsilon_{jl}} t_{jl} = 0$.

(3) If $\epsilon'_{ik} + \epsilon_{jl} = 2$, then $b^{\epsilon'_{ik}} \delta'_{ik} \delta_{jl} a^{\epsilon_{jl}} \in T$.

We define

$$h_{kl}^{(i,j)} = \begin{cases} \alpha_{ikjl} t_{jl} & \text{in case (1)}, \\ 0 & \text{in case (2)}, \\ b^{\epsilon'_{ik}} \delta'_{ik} \delta_{jl} a^{\epsilon_{jl}} t_{jl} & \text{in case (3)}. \end{cases}$$

Note that $h_{kl}^{(i,j)} \in T$ for all $1 \leq i, j \leq r$ and $1 \leq k, l \leq n$.

Suppose that T is locally nilpotent. Then there exists a positive integer m such that

$$\{h_{kl}^{(i,j)} \mid 1 \leq i, j \leq r \text{ and } 1 \leq k, l \leq n\}^m = 0.$$

Now if $i_1, \ldots, i_{m+1} \in \{1, \ldots, r\}$, we have

$$p_{i_1} \cdots p_{i_{m+1}} = \sum_{k_1, \ldots, k_{m+1}=1}^{n} \cdots t_{i_1 k_1} h_{k_1 k_2}^{(i_1, i_2)} h_{k_2 k_3}^{(i_2, i_3)} \cdots h_{k_m k_{m+1}}^{(i_m, i_{m+1})} \cdots = 0.$$

Thus I is locally nilpotent. ∎

Incidentally, if we assume F is uncountable and T is merely nil then one sees without difficulty that the Jacobson radical of the ring of [4, example 2.3] is nil.

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